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FINITE ELEMENT ANALYSIS OF THE STEADY NAVIER–STOKES EQUATIONS BY A MULTIPLIER METHOD

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SUMMARY

In this paper a fully explicit finite element method (FEFEM) is presented for solving steady incompressible viscous flow problems. This full explicitness is achieved by combining the multiplier (or augmented Lagrangian) method with a pseudo-time-iteration method. FEFEM needs no global matrix at all and is of great advantage to large-scale problems because they can be solved within the limit of core memory.

The optimum choice of a time increment and a penalty parameter is discussed and the driven cavity flow at a Reynolds number of 1000 is computed with a refined mesh (60 × 60 elements).

KEY WORDS Finite Elements Steady Flow Navier–Stokes equations Multiplier Methods Pseudo-Time-Iteration Method

INTRODUCTION

In recent years the finite element method (FEM) has been increasingly used for solving incompressible viscous flow problems. Nevertheless, applications of FEM to large-scale problems are very rare. This originates mainly in two reasons: one is the insufficient memory and calculating speed of computers, and the other—this is important—is a lack of formulations suitable for large-scale problems.

A well-known method which is applicable to large-scale problems is the fractional step method.[1–4] It has an efficient implicit pressure/explicit velocity algorithm and is suitable for transient calculations. However, it needs too much computation (CPU plus I/O) time to get a steady solution, because the Poisson equation for the pressure must be solved at each iterative stage. Therefore, it can be said that even the fractional step method is ineffective for solving large-scale steady problems.

In this paper a fully explicit finite element method (FEFEM) is presented for solving steady incompressible viscous flow problems. This full explicitness is achieved by combining the multiplier (or augmented Lagrangian) method with a pseudo-time-iteration method. The multiplier method is very suitable for solving steady incompressible viscous flow problems and many algorithms by this method have been already presented.[5–7] However, they are not fully explicit and need a global matrix. On the other hand, FEFEM is really fully explicit and needs, of course, no global matrix at all. This is of great advantage to large-scale problems because they can be solved within the limit of core memory.

In the next section the governing equations and the boundary conditions are presented. Then the multiplier method is briefly reviewed and the algorithm of FEFEM is presented. Next the

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discretization by the Galerkin method is presented and an approximate integration technique is introduced to save CPU time. Then the optimum choice of a time increment and a penalty parameter is discussed, and in the last section the driven cavity flow at a Reynolds number of 1000 is computed with a refined mesh (60 × 60 elements).

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The steady Navier–Stokes equations and the continuity equation are

$$\rho(\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \left(\kappa + \frac{\mu}{3}\right)\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u} \tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ is the velocity vector, p the pressure, ρ the density, μ the shear viscosity and κ the bulk viscosity, and the other mathematical symbols are used in the standard manner. The second term of the right-hand side of equation (1), which is usually omitted because of equation (2), plays a very important role in the multiplier method.

In the case of Stokes flows, equation (1) is reduced to

$$\nabla p - \eta\nabla(\nabla\cdot\mathbf{u}) - \mu\nabla^2\mathbf{u} = 0 \tag{3}$$

where

$$\eta \equiv \kappa + \frac{\mu}{3}\left(\geqslant \frac{\mu}{3}\right) \tag{4}$$

As for equation (3), the variational integral exists and is expressed as follows:

$$J(\mathbf{u}, p) \equiv \int_\Omega \{\tfrac{1}{2}\mu\nabla\mathbf{u}:(\nabla\mathbf{u})^{\mathrm{T}} - p\nabla\cdot\mathbf{u} + \tfrac{1}{2}\eta(\nabla\cdot\mathbf{u})^2\}\,\mathrm{d}\Omega \tag{5}$$

where Ω is a domain of the flow field.

The essential and the natural boundary conditions are obtained by calculating the first variation of equation (5), and they are expressed as follows:

$$\begin{aligned} \mathbf{u} &= \mathbf{a}, && \text{on } \Gamma_1 \\ (-p + \eta\nabla\cdot\mathbf{u})\mathbf{n} + \mu\frac{\partial\mathbf{u}}{\partial n} &= \mathbf{b}, && \text{on } \Gamma_2 \end{aligned} \tag{6}$$

where $\mathbf{a}$ and $\mathbf{b}$ are given boundary data, $\mathbf{n}$ is the unit vector outward normal to the boundary and $\partial/\partial n$ is the outward normal derivative to the boundary. Γ_1 and Γ_2 are subsets of whole boundary Γ and satisfy the following conditions:

$$\begin{aligned} \overline{\Gamma_1 \cup \Gamma_2} &= \Gamma \\ \Gamma_1 \cap \Gamma_2 &= \varnothing \end{aligned} \tag{7}$$

MULTIPLIER METHOD

When a functional $\int_\Omega f(u)\mathrm{d}\Omega$ and a function $h(\mathbf{u})$ are given, the problem

$$\text{Find } \mathbf{u} \text{ such that } \int_\Omega f(\mathbf{u})\,\mathrm{d}\Omega \text{ is minimized subject to } h(\mathbf{u}) = 0 \tag{8}$$

is a constrained minimization problem. In the multiplier method, in order to transform this constrained problem into a sequence of unconstrained problems, the augmented Lagrangian (9) is introduced:[8]

$$L(\mathbf{u},\xi) \equiv \int_\Omega f(\mathbf{u})\,\mathrm{d}\Omega - \int_\Omega \xi h(\mathbf{u})\,\mathrm{d}\Omega + \frac{\sigma}{2}\int_\Omega \{h(\mathbf{u})\}^2\,\mathrm{d}\Omega \tag{9}$$

where

ξ is a Lagrange multiplier

σ is a positive penalty parameter

By using the augmented Lagrangian (9), the algorithm of the multiplier method is given as follows:[8]

0. Give an arbitrary ξ^0 and set $m = 0$.
1. Find $\mathbf{u}^{m+1}$ such that:

$$\delta L(\mathbf{u}^{m+1}, \xi^m) = 0$$

2. If $h(\mathbf{u}^{m+1}) = 0$ then stop.
3. Calculate ξ^{m+1} by

$$\xi^{m+1} = \xi^m - \sigma h(\mathbf{u}^{m+1})$$

4. Set $m = m + 1$ and go to 1.

Now, substituting $\frac{1}{2}\mu\nabla\mathbf{u}{:}(\nabla\mathbf{u})^{\mathrm{T}}$, $\nabla\cdot\mathbf{u}$, p and η for $f(\mathbf{u})$, $h(\mathbf{u})$, ξ and σ, respectively, it is found that $L(\mathbf{u},\xi)$ amounts to $J(\mathbf{u},p)$. Consequently, the multiplier method is directly applicable to the Stokes flow problems. The algorithm is given as follows:

0. Give an arbitrary p^0 and set $m = 0$.
1. Find $\mathbf{u}^{m+1}$ such that:

$$\delta J(\mathbf{u}^{m+1}, p^m) = 0 \tag{10}$$

2. If $\nabla\cdot\mathbf{u}^{m+1} = 0$ then stop.
3. Calculate p^{m+1} by

$$p^{m+1} = p^m - \eta\nabla\cdot\mathbf{u}^{m+1} \tag{11}$$

4. Set $m = m + 1$ and go to 1.

For the Navier–Stokes equations (1) it is known that there exists no variational integral corresponding to $J(\mathbf{u},p)$.[9] The multiplier method, however, is also applicable to this case by regarding δJ as not the first variation but the weighted residual integral, i.e. by replacing equation (10) with

$$\int_\Omega \{\rho(\mathbf{u}^{m+1}\cdot\nabla)\mathbf{u}^{m+1} + \nabla p^m - \eta\nabla(\nabla\cdot\mathbf{u}^{m+1}) - \mu\nabla^2\mathbf{u}^{m+1}\}\cdot\mathbf{W}\,\mathrm{d}\Omega = 0 \tag{12}$$

where $\mathbf{W}$ is a weighting function which vanishes on Γ_1.[5,6]

As was said in the introduction, the aim of this paper is to construct a scheme which needs no global matrix at all. However, it is impossible to obtain $\mathbf{u}^{m+1}$ from equation (12) without solving the simultaneous equations.[5,6] To get rid of this contradiction the time derivative term $\rho(\partial\mathbf{u}/\partial t)$ is added to the left-hand side of equation (1) and it is approximated by forward finite difference. Considering the iteration number m to be the time step number, the following equation is obtained:

$$\int_\Omega \left\{\rho\frac{\mathbf{u}^{m+1} - \mathbf{u}^m}{\Delta t} + \rho(\mathbf{u}^m\cdot\nabla)\mathbf{u}^m + \nabla p^m - \eta\nabla(\nabla\cdot\mathbf{u}^m) - \mu\nabla^2\mathbf{u}^m\right\}\cdot\mathbf{W}\,\mathrm{d}\Omega = 0 \tag{13}$$

where Δt is a time increment. From equation (13) $\mathbf{u}^{m+1}$ is obtained without solving the simultaneous equations by using the lumped mass matrix. Consequently, the algorithm comes to the final form:

0. Give arbitrary $\mathbf{u}^0$ and p^0, and set $m=0$.
1. Find $\mathbf{u}^{m+1}$ by solving equation (13).
2. If $\nabla\cdot\mathbf{u}^{m+1}=0$ then stop.
3. Calculate p^{m+1} by equation (11).
4. Set $m=m+1$ and go to 1.

It should be noticed that the time derivative term is introduced only to obtain the steady solution (a kind of pseudo-time-iteration method), i.e. the converged solution is meaningful as the steady solution but the solutions at each time step are meaningless as the non-steady solutions.

DISCRETIZATIONS

In this section two-dimensional cases are treated. The four-node isoparametric elements are used, in which the velocity is interpolated by bilinear basis functions N_i and the pressure is piecewise constant.

In the following discussions, it is supposed that all the variables are suitably non-dimensionalized and that (u, v) denote the velocity components in an orthogonal Cartesian co-ordinate system (x, y).

The non-dimensional form of the pressure correction equation (11) is expressed as

$$p^{m+1}=p^m-\frac{\alpha}{R}\nabla\cdot\mathbf{u}^{m+1} \tag{14}$$

where R is the Reynolds number and α is a non-dimensional penalty parameter defined by

$$\alpha\equiv\frac{\eta}{\mu} \tag{15}$$

As the pressure is piecewise constant, the weighted residual integral of equation (14) is expressed as

$$p_e^{m+1}=p_e^m-\frac{\alpha}{R}D_e^{m+1} \tag{16}$$

$$D_e^{m+1}\equiv\frac{1}{\text{area }(\Omega_e)}\int_{\Omega_e}\nabla\cdot\mathbf{u}^{m+1}\,\mathrm{d}\Omega \tag{17}$$

where index e denotes an element number.

Equation (16) shows that p_e of each element plays the role of the Lagrange multiplier and the incompressibility constraint corresponding to p_e is given by

$$D_e=0 \tag{18}$$

Equation (18) represents the mass conservation of an element e. This means that the discrete divergence of the velocity has the same number of degrees of freedom as the discrete pressure, i.e. one per element. The Navier–Stokes equations must be discretized in consideration of this fact.

Applying the Galerkin method to equation (13), the following equation is obtained for u (nearly the same equation is obtained for v):

$$\sum_e\int_{\Omega_e}N_i\left(\frac{u^{m+1}-u^m}{\Delta t}\right)\mathrm{d}\Omega=\sum_e\int_{\Omega_e}\left(p_e^m-\frac{\alpha}{R}\nabla\cdot\mathbf{u}^m\right)\frac{\partial N_i}{\partial x}\mathrm{d}\Omega-\frac{1}{R}\sum_e\int_{\Omega_e}\nabla N_i\cdot\nabla u^m\,\mathrm{d}\Omega$$
$$-\sum_e\int_{\Omega_e}N_i(\mathbf{u}^m\cdot\nabla)u^m\,\mathrm{d}\Omega+\int_\Gamma N_i b_x\,\mathrm{d}\Gamma \tag{19}$$

where b_x is the x-component of the boundary condition $(6)_2$.

The first term of the right-hand side of equation (19) must be replaced with

$$\sum_e \int_{\Omega_e} \left(p_e^m - \frac{\alpha}{R} D_e^m \right) \frac{\partial N_i}{\partial x} \mathrm{d}\Omega \left(= \sum_e \left(p_e^m - \frac{\alpha}{R} D_e^m \right) \int_{\Omega_e} \frac{\partial N_i}{\partial x} \mathrm{d}\Omega \right) \tag{20}$$

because of the above discussion. This is the consistent treatment of the penalty term.[10,11]

The left-hand side of equation (19) yields the mass matrix. As was said in the previous section, the full explicitness of the algorithm is achieved by replacing the consistent mass matrix with the lumped one which is obtained by row-sum at element level. This approximation has no influence on the steady solution because the time derivative term vanishes at convergence.

In order to compute the numerical examples of this paper the following approximations are made to the diffusion (second) and the convection (third) terms of the right-hand side of equation (19):

1. $$\int_{\Omega_e} \frac{\partial u^m}{\partial x} \frac{\partial N_i}{\partial x} \mathrm{d}\Omega \rightarrow \left(\frac{\partial u^m}{\partial x} \right)_e \int_{\Omega_e} \frac{\partial N_i}{\partial x} \mathrm{d}\Omega, \text{ etc.} \tag{21}$$

2. $$\int_{\Omega_e} u^m \frac{\partial u^m}{\partial x} N_i \,\mathrm{d}\Omega \rightarrow u_e^m \left(\frac{\partial u^m}{\partial x} \right)_e \int_{\Omega_e} N_i \,\mathrm{d}\Omega, \text{ etc.} \tag{22}$$

where

$$\left(\frac{\partial u^m}{\partial x} \right)_e \equiv \frac{1}{\text{area } (\Omega_e)} \int_{\Omega_e} \frac{\partial u^m}{\partial x} \mathrm{d}\Omega; \quad u_e^m \equiv \frac{1}{\text{area } (\Omega_e)} \int_{\Omega_e} u^m \,\mathrm{d}\Omega \tag{23}$$

These approximations, which have nothing to do with the present method, are very usefull to save CPU time, because the right-hand sides of equations (21) and (22) can be calculated without using any numerical integration technique (see the Appendix). Of course there is a fear that these approximations may lower the accuracy of solutions, but the accuracy of solutions depends not only on the accuracy of integration but also on the mesh pattern and the size of individual elements. Generally speaking, if we want a highly accurate solution, we need the more CPU time. In practice we therefore should take the computation time into consideration and come to an understanding on fair terms. Some researchers employ the one-point Gaussian quadrature to minimize computation time.[3,12]

As for interpolations it is well known that the bilinear velocity and the piecewise constant pressure can exhibit a singular 'chequerboard' mode of the pressure field under certain types of boundary conditions,[13] and extensive researches have recently been done to study the relation between the incompressibility constraint and the pressure field.[14,15] However, it should be noted that for virtually all practical problems the chequerboard mode is either absent or can be filtered out by suitable smoothing techniques.[16–18] Therefore, it would be unnecessary to extend the method to more costly and complicated higher-order elements.

STABILITY CONDITIONS

It is generally said that the explicit method is conditionally stable and the time increment Δt has to satisfy the following conditions:[19]

$$0 < \Delta t \leqslant \Delta t_0 \tag{24}$$

where Δt_0 is the limiting value of Δt and is seen to be governed by various factors such as the Reynolds number, mesh size and, in this case, the penalty parameter α.

The Navier–Stokes equations are a generalization of the simpler convection–diffusion equation and, in one-dimensional cases, they are reduced to

$$\frac{u^{m+1}-u^m}{\Delta t} = -u^m \frac{\partial u^m}{\partial x} + \frac{1}{R}(\alpha+1)\frac{\partial^2 u^m}{\partial x^2} \tag{25}$$

where the pressure term is omitted.

There are two important parameters which govern the stability of equation (25), namely,

$$d \equiv \frac{\alpha+1}{R}\frac{\Delta t}{(\Delta h)^2}\text{: Diffusion number} \tag{26}$$

$$C \equiv \frac{|u^m|_{\max}\Delta t}{\Delta h}\text{:Courant number} \tag{27}$$

where Δh is a mesh size and $|u^m|_{\max}$ is the maximum absolute value of u^m.

The stability conditions of equation (25) depend on how one spatially discretizes equation (25). For example, in case of the centred-space finite differences on a regular grid, the stability conditions are given by[19]

$$d \leqslant \tfrac{1}{2} \quad \text{and} \quad c^2 \leqslant 2d \tag{28}$$

i.e.

$$\Delta t \leqslant \frac{R(\Delta h)^2}{2(\alpha+1)} \quad \text{and} \quad \Delta t \leqslant \frac{2(\alpha+1)}{R|u^m|^2_{\max}} \tag{29}$$

The conditions (28) are illustrated in Figure 1 together with (4).

In the case of FEM, and that in case of two or three-dimensional equations, the stability conditions must be different from (28), but yet they can be expected to be given by inequalities similar to (28). In order to verify it numerically, let us consider the step flow as a test example. The finite element mesh and the boundary conditions are shown in Figure 2. The Reynolds number is 50, which is based on the inlet width and the maximum velocity there.

The limiting values of time increment Δt against each penalty parameter α are plotted in Figure 3, and the stability conditions of this example are given approximately by

$$d \leqslant \tfrac{1}{3} \quad \text{and} \quad c^2 \leqslant 3d \tag{30}$$

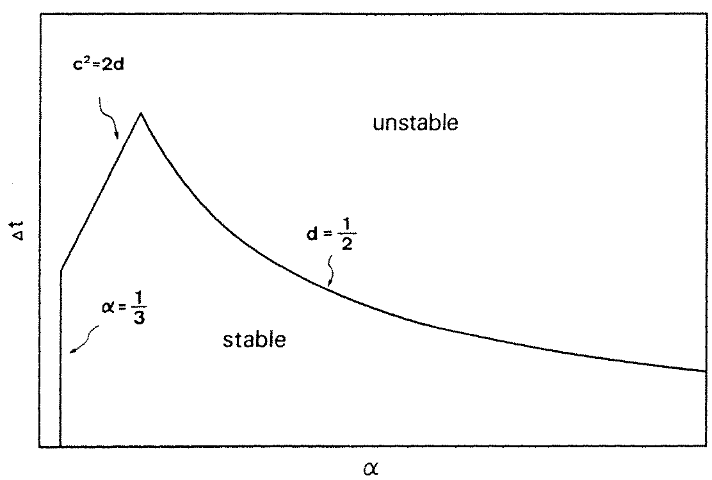

Figure 1. Neutral stability curve for one-dimensional centred-space finite difference equation

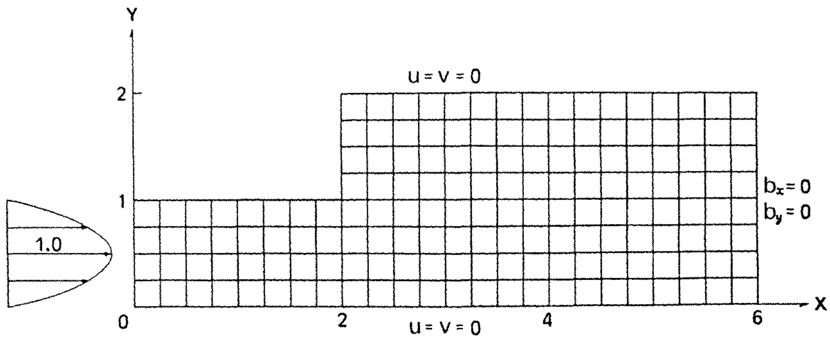

Figure 2. Finite element mesh and boundary conditions for step flow

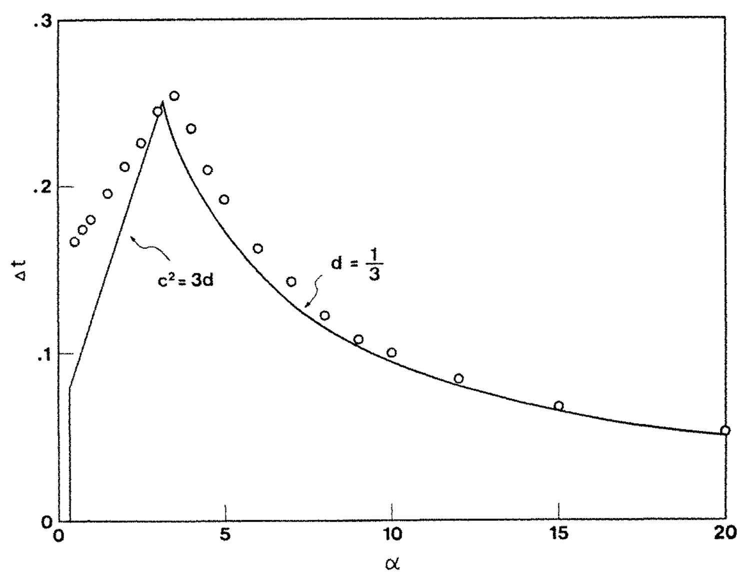

Figure 3. Stability conditions for step flow from numerical experiments

where $\Delta h = 0{\cdot}25$, $R = 50{\cdot}0$ and $|u^m|_{\max} = 1{\cdot}0$. It is found that the conditions (30) are very similar to (28). The computed velocity vectors and pressure contours are shown in Figure 4. It is needless to say that these computed results do not depend on those parameters.

The next problem is to find the optimum parameters Δt and α which maximize the convergence rate. This is, however, very difficult and maybe there is no rational way to estimate such optimum values *a priori*. They therefore should be determined by a numerical experiment. As an example the experimental results corresponding to the data in Figure 3 are shown in Figure 5, where the ordinate is the time step at which $\nabla \cdot \mathbf{u}$ becomes less than 0·001. From this Figure it is found that the optimum value of α for this example is between 4 and 5 and that the convergence rate becomes much worse as α becomes smaller than the optimum value.

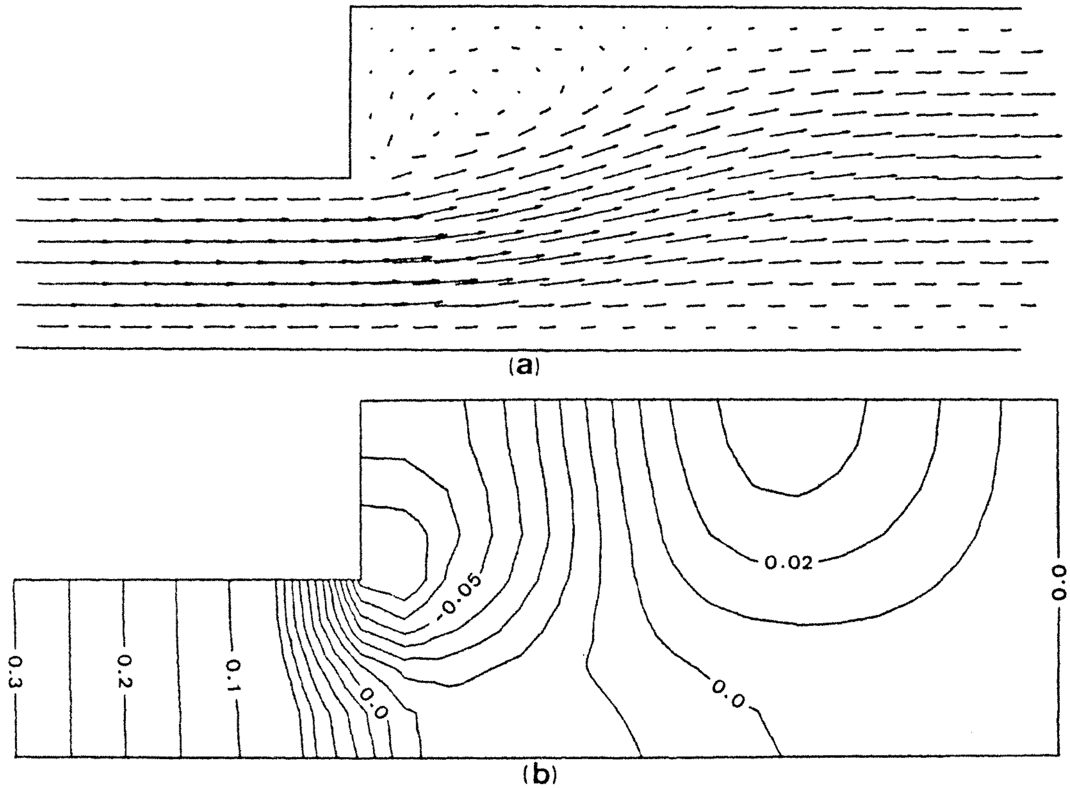

Figure 4. Step flow ($R = 50$): (a) velocity vectors; (b) pressure contours

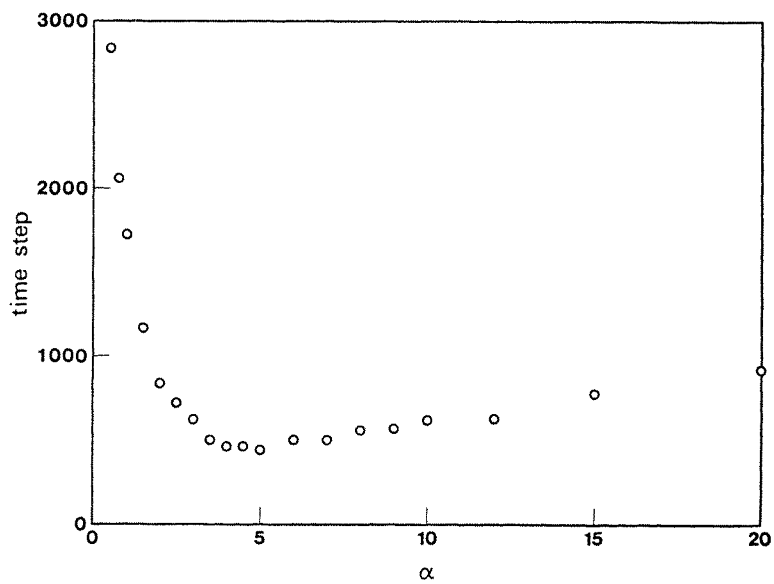

Figure 5. Convergence rate for step flow from numerical experiments

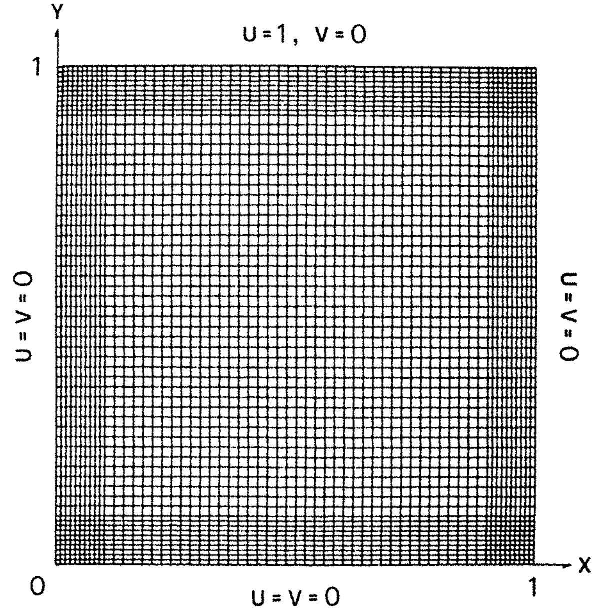

Figure 6. Finite element mesh and boundary conditions for driven cavity flow

NUMERICAL EXAMPLE

Driven cavity flow

The finite element mesh and the boundary conditions are shown in Figure 6. The computing conditions are as follows: $R = 1000$; $\Delta t = 0{\cdot}009$; $\alpha = 2{\cdot}7$; $\mathbf{\nabla}\cdot\mathbf{u} < 10^{-4}$ (stopping condition). The computation was performed in single precision (32 bits per word) on the NKK Technical Research Center IBM 308ID computer. The used memory size was only 460K bytes, the iteration number was 16,241 and CPU time was 85·64 min.

The computed velocity vectors, pressure contours, streamlines and vorticity contours are shown in Figure 7. The profiles of the horizontal velocity along the vertical centre line of the cavity ($x = 0{\cdot}5$) and the vertical velocity along the horizontal centre line of the cavity ($y = 0{\cdot}5$) are illustrated in Figure 8. They show almost exact agreement with those of Ghia *et al.* (uniform grid; 129 × 129).[20]

CONCLUSIONS

The fully explicit algorithm for solving the steady incompressible Navier–Stokes equations has been presented. This full explicitness was achieved by combining the multiplier method with the pseudo-time-iteration method. The present method really needs no global matrix at all and will open the way to large-scale, three-dimensional problems.

The choice of the optimum parameters is a problem left for the future.

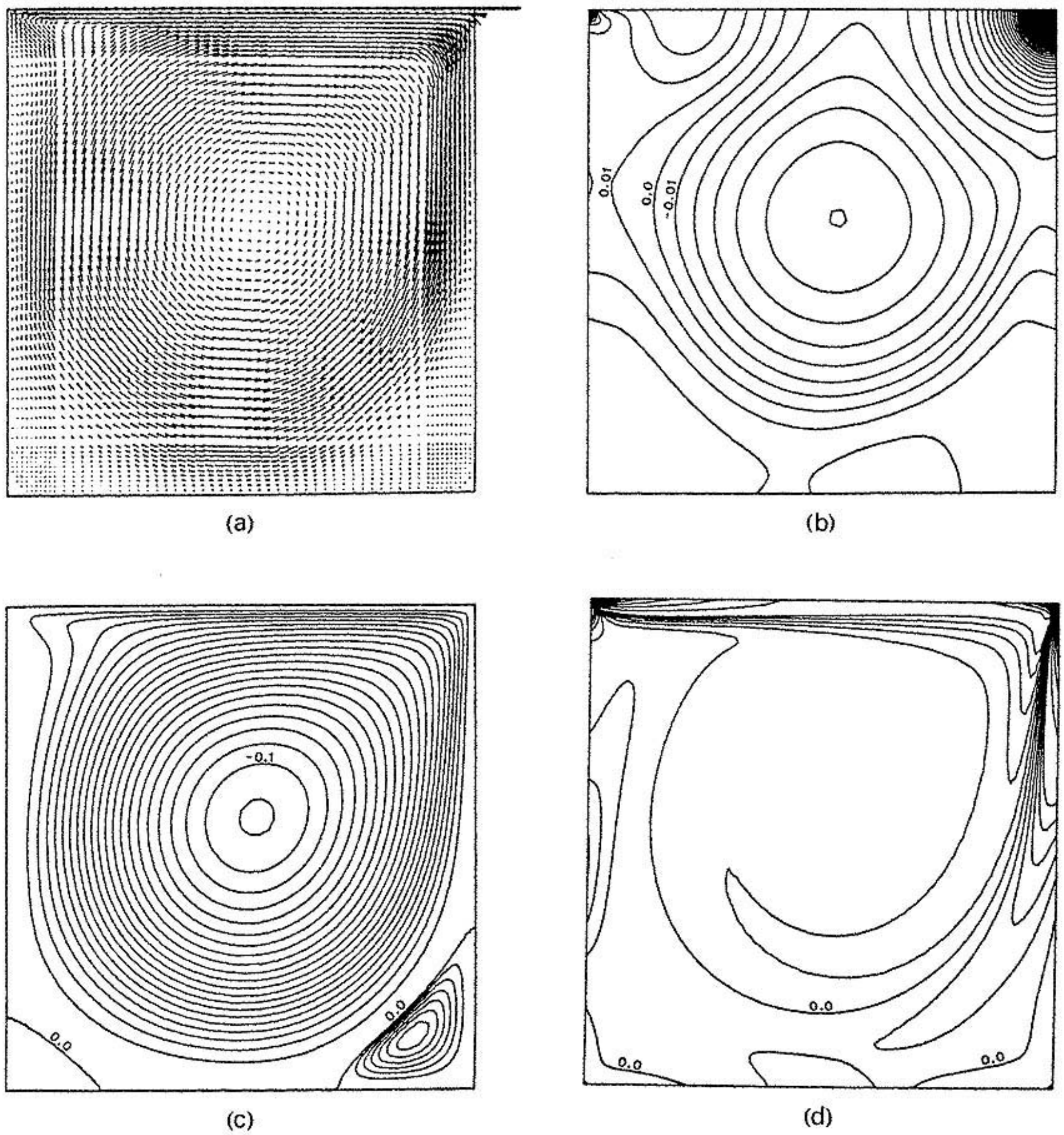

Figure 7. Driven cavity flow ($R = 1000$): (a) velocity vectors; (b) pressure contours; (c) streamlines; (d) vorticity contours

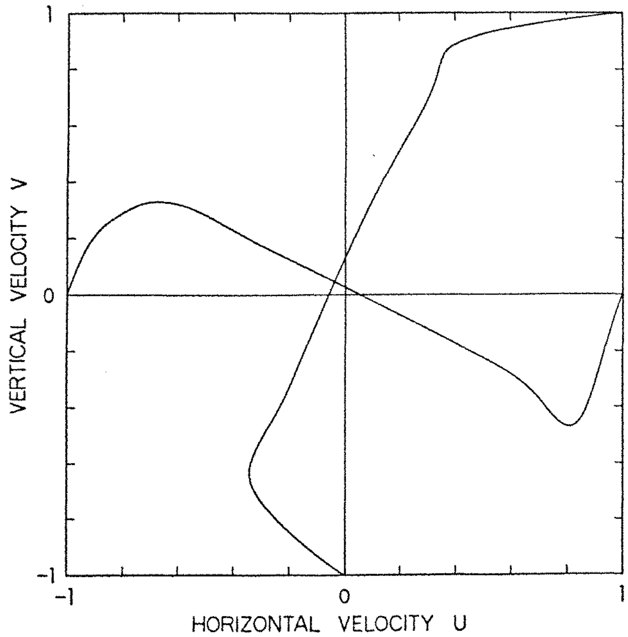

Figure 8. Velocity profiles along horizontal and vertical centre lines

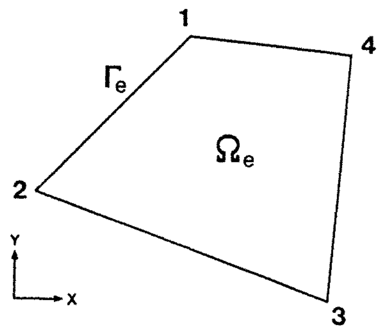

Figure 9. Typical four-node isoparametric element

APPENDIX: SOME INTEGRAL FORMULAE (SEE FIGURE 9)

1. $$\text{area}(\Omega_e)=\frac{1}{2}\left(\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_2 & y_2\\ 1 & x_3 & y_3\end{vmatrix}+\begin{vmatrix}1 & x_1 & y_1\\ 1 & x_3 & y_3\\ 1 & x_4 & y_4\end{vmatrix}\right)$$
$$=\tfrac{1}{2}\{(x_3-x_1)(y_4-y_2)-(y_3-y_1)(x_4-x_2)\}$$

2. $$\int_{\Omega_e}\frac{\partial u}{\partial x}\mathrm{d}\Omega=\int_{\Gamma_e}un_x\,\mathrm{d}\Gamma=\tfrac{1}{2}\{(u_3-u_1)(y_4-y_2)-(y_3-y_1)(u_4-u_2)\}$$
$$\int_{\Omega_e}\frac{\partial u}{\partial y}\mathrm{d}\Omega=\int_{\Gamma_e}un_y\,\mathrm{d}\Gamma=\tfrac{1}{2}\{(x_3-x_1)(u_4-u_2)-(u_3-u_1)(x_4-x_2)\}$$

3. $$\int_{\Omega_e} N_1\,\mathrm{d}\Omega = \frac{1}{8}\left(A - \frac{B}{3} + \frac{C}{3}\right), \quad \int_{\Omega_e} N_2\,\mathrm{d}\Omega = \frac{1}{8}\left(A - \frac{B}{3} - \frac{C}{3}\right)$$

$$\int_{\Omega_e} N_3\,\mathrm{d}\Omega = \frac{1}{8}\left(A + \frac{B}{3} - \frac{C}{3}\right), \quad \int_{\Omega_e} N_4\,\mathrm{d}\Omega = \frac{1}{8}\left(A + \frac{B}{3} + \frac{C}{3}\right)$$

where (n_x, n_y) are (x, y) components of the unit vector outward normal to the element boundary Γ_e, and A, B and C are given by

$$A \equiv (x_3 - x_1)(y_4 - y_2) - (y_3 - y_1)(x_4 - x_2)$$
$$B \equiv (x_3 - x_2)(y_4 - y_1) - (y_3 - y_2)(x_4 - x_1)$$
$$C \equiv (x_2 - x_1)(y_4 - y_3) - (y_2 - y_1)(x_4 - x_3)$$

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